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Operator Method of Nonstationary Temperature Field Calculation in Environment With an Isolated Cylindrical Pipe

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Abstract **– The following research is important for the understanding of thermal conductivity processes in the setting typical for thermal pumps and for bringing calculations nearer to the real situation. Considering a non-stationary process, the temperature field is described in the polar coordinate system by using Laplace's equation and corresponding mixed-type boundary data. The solution was obtained by the Laplace transform method, applying an integral function of complex variables. The inverse Laplace transform and the original temperature are expressed as an integral. For the integration, a closed contour, which rules out branching and provides an integral of an analytic function, is employed. Cauchy's theorem is applied to the calculations. As a result, indefinite integrals have been derived for temperature estimation in the heat pipe coating and the surrounding environment, depending on the fluid temperature within the heat pipe.**

Keywords **– Heat pumps, Laplace transform, thermal conductivity.**

I. Introduction

The following research is important for the understanding of thermal conductivity processes in the setting typical for a modern renewable heat source with a further development perspective, namely, thermal pumps. The understanding of the limitations of the thermal pumps technology also has very important economic implications. Imperfections in thermal conductivity may change the efficiency of thermal pumps technology several times and should be taken into account when taking investment decisions. Since the general principle of heat transfer in such circumstances is similar in other settings as well, the following research is also useful with other applications. The heat flows currently calculated may significantly differ from the actual values, a difference that is very well observable between theoretical calculations of ground heat pump productivity and actually observed heat flows. The temperature field parameter methodology developed within this study is needed to determine the heat flow with a greater degree of accuracy.

II. Description of Non-Stationary Temperature Field

A non-stationary temperature field $T(r,\tau)$ [K] in environment with a cylindrical pipe (Fig. 1), which is filled with a special fluid, at a constant temperature T_0 [K], can be described [10] with a thermal conductivity equation in polar coordinates:

$$\frac{\partial T(r,\tau)}{\partial \tau} = a\left(\frac{\partial^2 T(r,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,\tau)}{\partial r}\right);\ r \ge r_0;\ \tau \ge 0, \quad (1)$$

where

$$\partial T(r,\tau) = \begin{cases} T_0(r,\tau),\ \mathit{if}\ \ 0 \le r \le r_0; & \text{fluid in pipe;} \\ T_1(r,\tau),\ \mathit{if}\ \ r_0 \le r \le r_1; & \text{in pipe housing;} \\ T_2(r,\tau),\ \mathit{if}\ \ r_1 \le r \le r_2; & \text{in pipe insulation;} \\ T_3(r,\tau),\ \mathit{if}\ \ r \le r_2; & \text{in } soil. \end{cases} \quad (2)$$

a – thermal diffusion coefficient, $\mathrm{m^2/s}$;

$$a = \begin{cases} a_1, & \text{if}\ \ r_0 \le r \le r_1; \\ a_2, & \text{if}\ \ r_1 \le r \le r_2; \\ a_3, & \text{if}\ \ \ \ \ \ \ \ r \ge r_2. \end{cases} \quad (3)$$

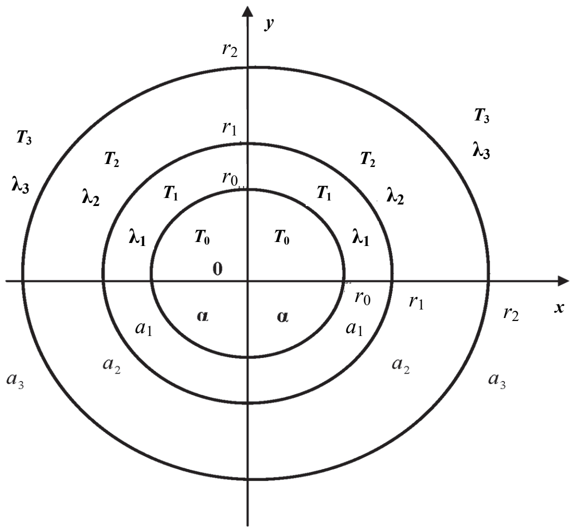

Fig. 1. Cross-section of a pipe.

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Equation (1) is overwritten using (2) and (3):

$$\frac{\partial T_i(r,\tau)}{\partial\tau} = a_i\left(\frac{\partial^2 T_i(r,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial T_i(r,\tau)}{\partial r}\right), i = 1;\ 2;\ 3. \quad (4)$$

On boundary $r = r_0$, heat exchange between the fluid and the pipe housing proceeds in accordance with Newton's law:

$$\lambda_1\frac{\partial T_1(r_0,\tau)}{\partial r} = -\alpha\left[T_0 - T_1(r_0,\tau)\right], \quad (5)$$

where α – heat transfer coefficient, W/(m^2 K);
T_0 – fluid temperature in pipe, K.

On boundary $r = r_1$, the temperatures and heat flows must be equal:

$$T_2(r_1,\tau) = T_1(r_1,\tau); \quad (6)$$

$$\lambda_2\frac{\partial T_2(r_1,\tau)}{\partial r} = \lambda_1\frac{\partial T(r_1,\tau)}{\partial r}, \quad (7)$$

where λ_1 – pipe housing heat transfer coefficient, W/(m K).

On boundary $r = r_2$, the temperatures and heat flows must be equal:

$$T_2(r_2,\tau) = T_3(r_2,\tau); \quad (8)$$

$$\lambda_2\frac{\partial T_2(r_2,\tau)}{\partial r} = \lambda_3\frac{\partial T(r_2,\tau)}{\partial r}, \quad (9)$$

where λ_2 – pipe insulation heat transfer coefficient, W/(m K);
λ_3 – soil heat transfer coefficient, W/(m K).

Temperature of the starting moment $\tau = 0$:

$$T_i(r,0) = T_{0i}; \qquad i = 1, 2, 3. \quad (10)$$

At a large distance from the pipe axis, the temperature is constant:

$$T_3(\infty,\tau) = T_\infty \quad (11)$$

and

$$\frac{\partial T_3(\infty,\tau)}{\partial r} = 0. \quad (12)$$

It can be assumed that

$$T_{01} = T_{02} = T_{03} = T_\infty. \quad (13)$$

Using dimensionless variables and parameters [1]:

$$\frac{r}{r_0} = \rho; \quad \frac{r_1}{r_0} = \rho_1; \quad \frac{r_2}{r_0} = \rho_2; \quad (14)$$

$$\frac{T_i}{T_0} = \Theta_i; \quad \frac{T_{0i}}{T_0} = \Theta_{0i}; \quad \frac{T_\infty}{T_0} = \Theta_\infty = \Theta_{0i}; i = 1, 2, 3; \quad (15)$$

$$\frac{a_1}{a_2} = k_2^2; \quad \frac{a_1}{a_3} = k_3^2; \quad \frac{\lambda_1}{\lambda_2} = \Lambda_1; \quad \frac{\lambda_2}{\lambda_3} = \Lambda_2; \quad (16)$$

$$F = \frac{a_1\tau}{r_0^2}; \quad Bi = \frac{\alpha r_0}{\lambda_1}. \quad (17)$$

$$\Theta(p,F) = \begin{cases} 1, \text{ if } 0 \le \rho \le 1; & \text{fluid in pipe;} \\ \Theta_1(\rho,F), \text{ if } 1 \le \rho \le \rho_1; & \text{in pipe housing;} \\ \Theta_2(\rho,F), \text{ if } \rho_1 \le \rho \le \rho_2; & \text{in pipe insulation;} \\ \Theta_3(\rho,F), \text{ if } \rho \le \rho_2; & \text{in soil.} \end{cases} \quad (18)$$

The task is solved using Laplace transformation [1]–[5]:

$$\bar{\Theta}_i(\rho,p) = \int_0^{+\infty} e^{-pF}\Theta_i(\rho,F)\,\mathrm{d}F \quad (19)$$

and temperature images are obtained:

$$\bar{\Theta}_1(\rho,p) = C_1 I_0\left(\rho\sqrt{p}\right) + C_2 K_0\left(\rho\sqrt{p}\right) + \frac{\Theta_{01}}{p}; \quad 1 \le \rho \le \rho_1; \quad (20)$$

$$\bar{\Theta}_2(\rho,p) = C_3 I_0\left(k_2\rho\sqrt{p}\right) + C_4 K_0\left(k_2\rho\sqrt{p}\right) + \frac{\Theta_{02}}{p}; \rho_1 \le \rho \le \rho_2; \quad (21)$$

$$\bar{\Theta}_3(\rho,p) = C_5 K_0\left(k_3\rho\sqrt{p}\right) + \frac{\Theta_{03}}{p}; \rho_2 \le \rho < +\infty, \quad (22)$$

where $I_0(z)$ and $K_0(z)$ – modified zero-order Bessel (McDonald) functions of the first and second type [6]–[9].

Using the boundary conditions, the coefficients of the temperature images of the obtained expression are calculated (20)–(22):

$$\bar{\Theta}_1(\rho,p) = \frac{\Theta_{01}}{p} + \frac{\Delta C_1\left(\sqrt{p}\right)}{\Delta\left(\sqrt{p}\right)} I_0\left(\rho\sqrt{p}\right) + \frac{\Delta C_2\left(\sqrt{p}\right)}{\Delta\left(\sqrt{p}\right)} K_0\left(\rho\sqrt{p}\right); \quad (23)$$

$$\bar{\Theta}_2(\rho,p) = \frac{\Theta_{02}}{p} + \frac{\Delta C_3\left(\sqrt{p}\right)}{\Delta\left(\sqrt{p}\right)} I_0\left(k_2\rho\sqrt{p}\right) + \frac{\Delta C_4\left(\sqrt{p}\right)}{\Delta\left(\sqrt{p}\right)} K_0\left(k_2\rho\sqrt{p}\right); \quad (24)$$

$$\bar{\Theta}_3(\rho,p)=\frac{\Theta_{03}}{p}+\frac{\Delta C_5(\sqrt{p})}{\Delta(\sqrt{p})}K_0(k_3\rho\sqrt{p}), \tag{25}$$

where

$$\begin{aligned}\Delta(\sqrt{p})&=k_2k_3K_1(d)[I_1(b)K_0(c)+K_1(b)I_0(c)]\\&\cdot[\sigma_1(\sqrt{p})K_0(a)+\sigma_2(\sqrt{p})I_0(a)]\\&+\Lambda_1k_3K_1(d)[I_0(b)K_0(c)-K_0(b)I_0(c)]\\&\cdot[\sigma_1(\sqrt{p})K_1(a)-\sigma_2(\sqrt{p})I_1(a)]\\&+\Lambda_2k_2^2K_0(d)[I_1(b)K_1(c)-K_1(b)I_1(c)]\\&\cdot[\sigma_1(\sqrt{p})K_0(a)+\sigma_2(\sqrt{p})I_0(a)]\\&+\Lambda_1\Lambda_2k_2K_0(d)[I_0(b)K_1(c)+K_0(b)I_1(c)]\\&\cdot[\sigma_1(\sqrt{p})K_1(a)-\sigma_2(\sqrt{p})I_1(a)],\end{aligned} \tag{26}$$

$$\begin{aligned}\Delta C_1(\sqrt{p})&=\frac{Bi}{p}(\Theta_{01}-1)\Big[\;k_2k_3K_0(\bar{a})K_1(d)\\&\cdot[I_1(b)K_0(c)+K_1(b)I_0(c)]+\Lambda_1k_3K_1(\bar{a})K_1(d)\\&\cdot[I_0(b)K_0(c)-K_0(b)I_0(c)]+\Lambda_2k_2^2K_0(\bar{a})K_0(d)\\&\cdot[I_1(b)K_1(c)-K_1(b)I_1(c)]+\Lambda_1\Lambda_2k_2K_1(\bar{a})K_0(d)\\&\cdot[I_0(b)K_1(c)+K_0(b)I_1(c)]\;\Big];\end{aligned} \tag{27}$$

$$\begin{aligned}\Delta C_2(\sqrt{p})&=\frac{Bi}{p}(\Theta_{01}-1)\Big[\;-k_2k_3I_0(\bar{a})K_1(d)\\&\cdot[I_1(b)K_0(c)+K_1(b)I_0(c)]+\Lambda_1k_3I_1(\bar{a})K_1(d)\\&\cdot[I_0(b)K_0(c)-K_0(b)I_0(c)]-\Lambda_2k_2^2I_0(\bar{a})K_0(d)\\&\cdot[I_1(b)K_1(c)-K_1(b)I_1(c)]+\Lambda_1\Lambda_2k_2I_1(\bar{a})K_0(d)\\&\cdot[I_0(b)K_1(c)+K_0(b)I_1(c)]\;\Big];\end{aligned} \tag{28}$$

$$\begin{aligned}\Delta C_3(\sqrt{p})&=\frac{Bi}{p}(\Theta_{01}-1)\frac{\Lambda_1}{\rho_1\sqrt{p}}\\&\cdot[k_3K_0(c)K_1(d)+\Lambda_2k_2K_0(d)K_1(c)];\end{aligned} \tag{29}$$

$$\begin{aligned}\Delta C_4(\sqrt{p})&=\frac{Bi}{p}(\Theta_{01}-1)\frac{\Lambda_1}{\rho_1\sqrt{p}}\\&\cdot[k_3I_0(c)K_1(d)-\Lambda_2k_2K_0(d)I_1(c)];\end{aligned} \tag{30}$$

$$\Delta C_5(\sqrt{p})=\frac{Bi}{p}(\Theta_{01}-1)\frac{\Lambda_1\Lambda_2}{\rho_1^2 p}, \tag{31}$$

where

$\bar{a}=\rho_1\sqrt{p}$; $\quad b=k_2\rho_1\sqrt{p}$;

$$c=k_2\rho_2\sqrt{p};\qquad d=k_3\rho_2\sqrt{p}; \tag{32}$$

$$\sigma_1(\sqrt{p})=\sqrt{p}I_1(\sqrt{p})-BiI_0(\sqrt{p}); \tag{33}$$

$$\sigma_2(\sqrt{p})=\sqrt{p}K_1(\sqrt{p})+BiK_0(\sqrt{p}). \tag{34}$$

The original temperature is expressed [1]–[3] with the following integral:

$$\Theta_i(\rho,F)=\frac{1}{2\pi j}\int_{\gamma-j\omega}^{\gamma+j\omega}\mathrm{e}^{pF}\bar{\Theta}_i(\rho,p)\,\mathrm{d}p, \tag{35}$$

using formulas (23)–(25).

III. Pipe Housing Temperature Calculation

The original pPipe housing temperature is expressed in [1]–[3] using formulas (23) and (35) with an integral as follows:

$$\begin{aligned}\Theta_1(\rho,F)&=\Theta_{0i}+\frac{1}{2\pi j}\int_{\gamma-j\omega}^{\gamma+j\omega}\mathrm{e}^{pF}\frac{\Delta C_1(\sqrt{p})}{\Delta(\sqrt{p})}I_0(\rho\sqrt{p})\mathrm{d}p\\&+\frac{1}{2\pi j}\int_{\gamma-j\omega}^{\gamma+j\omega}\mathrm{e}^{pF}\frac{\Delta C_2(\sqrt{p})}{\Delta(\sqrt{p})}K_0(\rho\sqrt{p})\mathrm{d}p,\end{aligned} \tag{36}$$

where $j=\sqrt{-1}$.

The calculation of complex-variable integrals is discussed in monographs [1]–[3] and publications [4], [5].

Expressing complex-variable function integrals in a realistic manner, dimensionless expression of pipe housing temperature $(1\le\rho\ge\rho_1)$ yields the following equation:

$$\begin{aligned}\Theta_1(\rho,F)&=1+\frac{2}{\pi}Bi(\Theta_{01}-1)\\&\cdot\int_0^{+\infty}\mathrm{e}^{-u^2F}\frac{\phi(u)\psi_1(u)-\psi(u)\phi_1(u)}{\phi^2(u)+\psi^2(u)}J_0(\rho u)\frac{\mathrm{d}u}{u}-\frac{2}{\pi}Bi\\&\cdot(\Theta_{01}-1)\int_0^{+\infty}\mathrm{e}^{-u^2F}\frac{\phi(u)\psi_2(u)-\psi(u)\phi_2(u)}{\phi^2(u)+\psi^2(u)}K_0(\rho u)\frac{\mathrm{d}u}{u},\end{aligned} \tag{37}$$

where

$$\begin{aligned}\phi(u)&=k_2k_3Y_1(d)\cdot U_1(u)\cdot S_0(u)\\&+\Lambda_1k_3Y_1(d)\cdot U_2(u)S_1(u)+\Lambda_2k_2^2Y_0(d)\cdot U_3(u)\cdot S_0(u)\\&+\Lambda_1\Lambda_2k_2Y_0(d)\cdot U_4(u)\cdot S_1(u).\end{aligned} \tag{38}$$

$$\begin{aligned}\psi(u)&=k_2k_3J_1(d)\cdot U_1(u)\cdot S_0(u)+\Lambda_1k_3J_1(d)\cdot U_2(u)\cdot S_1(u)\\&+\Lambda_2k_2^2J_0(d)\cdot U_3(u)\cdot S_0(u)+\Lambda_1\Lambda_2k_2J_0(d)\cdot U_4(u)\cdot S_1(u);\end{aligned} \tag{39}$$

$$\begin{aligned}\phi_1(u)&=k_2k_3Y_0(\bar{a}_1)Y_1(d)\cdot U_1(u)\\&+\Lambda_1k_3Y_1(\bar{a}_1)Y_1(d)\cdot U_2(u)\;\;\Lambda_2k_2^2Y_0(\bar{a}_1)Y_0(d)\cdot U_3(u)\end{aligned}$$

$$+\Lambda_1\Lambda_2 k_2 Y_1(\bar{a}_1)Y_0(d)\cdot U_4(u); \quad (40)$$

$$\psi_1(u)=k_2k_3Y_0(\bar{a}_1)J_1(d)\cdot U_1(u)$$
$$+\Lambda_1 k_3 Y_1(\bar{a}_1)J_1(d)\cdot U_2(u)+\ \Lambda_2 k_2^2 Y_0(\bar{a}_1)J_0(d)U_3(u)$$
$$+\Lambda_1\Lambda_2 k_2 Y_1(\bar{a}_1)J_0(d)\cdot U_4(u); \quad (41)$$

$$\phi_2(u)=k_2k_3J_0(\bar{a}_1)Y_1(d)\cdot U_1(u)$$
$$+\Lambda_1 k_3 J_1(\bar{a}_1)Y_1(d)\cdot U_2(u)$$
$$+\Lambda_2 k_2^2 J_0(\bar{a}_1)Y_0(d)U_3(u)$$
$$+\Lambda_1\Lambda_2 k_2 J_1(\bar{a}_1)Y_0(d)\cdot U_4(u); \quad (42)$$

$$\psi_2(u)=k_2k_3J_0(\bar{a}_1)J_1(d)\cdot U_1(u)$$
$$+\Lambda_1 k_3 J_1(\bar{a}_1)J_1(d)\cdot U_2(u)$$
$$+\Lambda_2 k_2^2 J_0(\bar{a}_1)J_0(d)U_3(u)$$
$$+\Lambda_1\Lambda_2 k_2 J_1(\bar{a}_1)J_0(d)\cdot U_4(u); \quad (43)$$

where

$$\bar{a}_1=\rho_1 u; \quad b_1=k_2\rho_1 u; \quad c_1=k_2\rho_2 u; \quad d_1=k_3\rho_2 u; \quad (44)$$

$$s_1(u)=-uJ_1(u)-BiJ_0(u); \quad (45)$$

$$s_2(u)=-uY_1(u)-BiY_0(u); \quad (46)$$

$$S_0(u)=s_1(u)Y_0(\bar{a}_1)-s_2(u)J_0(\bar{a}_1); \quad (47)$$

$$S_1(u)=s_1(u)Y_1(\bar{a}_1)-s_2(u)J_1(\bar{a}_1); \quad (48)$$

$$U_1(u)=Y_1(b\ )J_0(c_1)-J_1(b_1)Y_0(c_1); \quad (49)$$

$$U_2(u)=J_0(b_1)Y_0(c_1)-Y_0(b_1)J_0(c_1); \quad (50)$$

$$U_3(u)=Y_1(b_1)J_1(c_1)-J_1(b_1)Y_1(c_1); \quad (51)$$

$$U_4(u)=J_0(b_1)Y_1(c_1)-Y_0(b_1)J_1(c_1). \quad (52)$$

Here $J_0(z),Y_0(z),J_1(z),Y_1(z)$ – zero-order and first-order order Bessel functions of the first and second type [6]–[9].

Calculating the non-real integrals, which are contained in formula (37), dimensionless temperature of the pipe housing is obtained.

IV. Pipe Housing Insulation Temperature Calculation

The original pipe housing insulation temperature is expressed in [1]–[3] using formulas (24) and (35) with an integral as follows:

$$\Theta_2(\rho,F)=\Theta_{02}+\frac{1}{2\pi j}\int_{\gamma-j\omega}^{\gamma+j\omega} e^{pF}\frac{\Delta C_3(\sqrt{p})}{\Delta(\sqrt{p})}I_0(k_2\rho\sqrt{p})dp$$

$$+\frac{1}{2\pi j}\int_{\gamma-j\omega}^{\gamma+j\omega} e^{pF}\frac{\Delta C_4(\sqrt{p})}{\Delta(\sqrt{p})}K_0(k_2\rho\sqrt{p})dp. \quad (53)$$

The calculation of complex-variable integrals is discussed in monographs [1]–[3] and publications [9], [10].

Expressing complex-variable function integrals in a realistic manner, with dimensionless expression of pipe insulation temperature $(\rho_1\le\rho\le\rho_2)$ the following formula is obtained:

$$\Theta_2(\rho,F)=1+(\Theta_{02}-\Theta_{01})-\left(\frac{2}{\pi}\right)^2(\Theta_{01}-1)\Lambda_1\frac{Bi}{\rho_1^2}$$
$$\cdot\left[\int_0^{+\infty} e^{-u^2F}\frac{\phi(u)\psi_3(u)+\psi(u)\phi_3(u)}{\phi^2(u)+\psi^2(u)}J_0(k_2\rho u)\frac{du}{u^2}\right. \quad (54)$$
$$\left.-\int_0^{+\infty} e^{-u^2F}\frac{\phi_4(u)J_0(\rho k_2 u)-\psi_4(u)Y_0(\rho k_2 u)}{\phi^2(u)+\psi^2(u)}J_0(k_2\rho u)\frac{du}{u^2}\right].,$$

where

$$\phi_3(u)=k_3\left[J_0(c_1)J_1(d_1)-Y_0(c_1)Y_1(d_1)\right]$$
$$+\Lambda_2 k_2\left[J_1(c_1)J_0(d_1)-Y_1(c_1)Y_0(d_1)\right]; \quad (55)$$

$$\psi_3(u)=k_3\left[Y_0(c_1)J_1(d_1)+J_0(c)_1Y_1(d_1)\right]$$
$$+\Lambda_2 k_2\left[J_1(c_1)Y_0(d_1)+Y_1(c_1)J_0(d_1)\right]; \quad (56)$$

$$\phi_4(u)=k_3J_0(\bar{a}_1)Y_1(d_1)+\Lambda_2k_2J_1(c_1)Y_0(d_1); \quad (57)$$

$$\psi_4(u)=k_3J_0(\bar{a}_1)J_1(d_1)+\Lambda_2k_2J_1(c_1)J_0(d_1). \quad (58)$$

Calculating the improper integrals in formula (54), the dimensionless temperature of the pipe insulation is obtained.

V. Soil Temperature Calculation

The original environment temperature is expressed in [1]–[3] using formulas (25) and (35) with an integral as follows:

$$\Theta_3(\rho,F)=\Theta_{03}$$
$$+\frac{1}{2\pi j}\int_{\gamma-j\omega}^{\gamma+j\omega} e^{pF}\frac{\Delta C_5(\sqrt{p})}{\Delta(\sqrt{p})}K_0(k_3\rho\sqrt{p})dp. \quad (59)$$

The integral in formula (59) is calculated as described above.

Expressing complex-variable function integrals in a realistic manner, with dimensionless expression of environment temperature $(\rho\ge\rho_2)$ the following formula is obtained:

$$\Theta_3(\rho,F)=\Theta_{03}+(1-\Theta_{01})\frac{\rho_1}{\rho_2}+\left(\frac{2}{\pi}\right)^3(\Theta_{01}-1)\Lambda_1\Lambda_2$$
$$\cdot\frac{Bi}{\rho_2^2}\int_0^{+\infty} e^{-u^2F}\frac{\phi(u)J_0(k_3\rho u)-\psi(u)Y_0(k_3\rho u)}{\phi^2(u)+\psi^2(u)}\frac{du}{u^3}. \quad (60)$$

Calculating the non-real integrals in formula (54), environmental dimensionless environment temperature is obtained.

VI. Conclusion

As there is no unified widely accepted methodology for the determination of heat flow between a heat source including the surrounding environment and the heat/cold user, the methodology offered is intended for bringing heat flow calculations nearer to the actual situation. This methodology can be used by heat technology specialists for designing, planning and other purposes.

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